Problem 1: Debye model in $D = 1, 2, 3$ dimensions  

For soundwaves (without dispersion) following relation is valid in all dimensions:

$$\omega_{l,t} = c_{l,t} \cdot k,$$  

$$k = \left| \vec{k} \right| = \frac{2\pi}{\lambda}$$

with frequency $\omega$, wave number $k$ and phase velocity $c_{l,t} = \text{const}$ for longitudinal and transverse waves. (For $D = 1$, only longitudinal waves exist.) Periodic boundary conditions in a cubic area $L^D$ are limiting the allowed $k$ values to:

$$\vec{k} = \frac{2\pi}{\lambda} (n_1, \ldots, n_D), \ n_i \in \mathbb{Z}$$

resulting in a volume of $\left( \frac{2\pi}{\lambda} \right)^D$ for every point in $k$-space. Calculate for $D = 1, 2, 3$

a) the frequency density (phonon density of states) $\rho(\omega)$ and include a sketch.

b) the limit $\omega_{\text{max}}$, which gives the maximum excitation frequency in the Debye-integral. Keep in mind that for $D$ dimensions $D$ different polarisations are possible.

c) a general expression for the total energy $U(T)$ and for the specific heat $c_V(T)$,

d) a qualitative temperature dependency of $c_V(T)$ for $T \to 0$

e) the high-temperature limit of $c_V(T)$ ($T \to \infty$). Compare this with the value obtained from the equipartition principle.

Problem 2: Unified dispersion relations

In the Debye model a phonon dispersion relation according to $\omega(\vec{k}) = c_s \cdot |\vec{k}|$ with sound velocity $c_s = \text{const}$ is used. Find for a dispersion relation according to $\omega(\vec{k}) = c_s \cdot |\vec{k}|^n$

a) the specific heat of a three-dimensional solid body for $T \to 0$

b) the exponent of the temperature dependency of the specific heat for $T \to 0$

Problem 3: Heat conduction

A house with quadratic base area ($10 \, \text{m} \times 10 \, \text{m}$) and $30^\circ$ roof ramp has the following insulation values:

$$K_1 = 1.30 \, \text{W/m-K} \text{ (attic floor, 10 cm concrete)}$$

$$K_2 = 0.50 \, \text{W/m-K} \text{ roof, 2 cm rooftiles)}$$

a) What is the attic temperature in the stationary case, assuming $20^\circ\text{C}$ room temperature and $0^\circ\text{C}$ outside temperature?

b) What is the thermal transfer $Q$?

c) $Q$ should be reduced to $10\%$ of the value calculated in b) by applying insulation material ($K_0 = 0.06 \, \text{W/mK}$) at the inner sides of the roof. Calculate the required thickness of the material.

d) Which amount of fuel oil ($\rho = 0.85 \, \text{g/cm}^3$, calorific value $10000 \, \text{kcal/kg}$) can be saved per day with this insulation?
Problem 4: Van Hove singularities

An optical phonon branch has (in three dimensions) the shape \( \omega(q) = \omega_0 - Aq^2 \) for \( q \) around 0. Show that:

\[
\rho(\omega) = \begin{cases} 
(L/2\pi)^3 \cdot (2\pi/A^{3/2}) \cdot (\omega - \omega_0)^{1/2} & \omega_0 < \omega \\
0 & \omega > \omega_0 
\end{cases}
\]

What effect does this have for the specific heat?